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Commutator relations reveal solvable structures in unambiguous state discrimination

M Kleinmann¹, H Kampermann¹, Ph Raynal² and D Bruß¹¹ Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany² Quantum Information Theory Group, Institut für Theoretische Physik I, and Max-Planck-Forschungsgruppe, Institut für Optik, Information und Photonik, Universität Erlangen-Nürnberg, D-91058 Erlangen, GermanyE-mail: kleinmann@thphy.uni-duesseldorf.de

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Online at stacks.iop.org/JPhysA/40/F871**Abstract**

We present a criterion, based on three commutator relations, that allows us to decide whether two self-adjoint matrices with non-overlapping support are simultaneously unitarily similar to quasi-diagonal matrices, i.e., whether they can be simultaneously brought into a diagonal structure with (2×2) -dimensional blocks. Application of this criterion to unambiguous state discrimination provides a systematic test whether the given problem is reducible to a solvable structure. As an example, we discuss unambiguous state comparison.

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1. Introduction

The commutator of two self-adjoint operators, which act on a Hilbert space, is a fundamental concept in quantum mechanics: two observables can be measured without uncertainty if and only if their commutator vanishes. This physical interpretation is connected to the mathematical fact that two Hermitian matrices can be diagonalized simultaneously if and only if their commutator is zero. A natural question to ask is when two Hermitian matrices can be simultaneously brought into a block-diagonal structure with blocks of the lowest non-trivial size, namely size 2×2 . Such structures are known as quasi-diagonal form and criteria for existence have been studied in [1, 2]: Watters [1] showed that a family of normal matrices can be simultaneously brought into a quasi-diagonal form if and only if each member of the family commutes with the squared commutator of an element of the family with any element from the algebra generated by the family. (Thus, testing this criterion requires us to show that infinitely many commutators vanish.) Laffey [2] studied a family with two members only. He showed that when the matrices in the family are positive semi-definite, they are simultaneously unitarily similar to quasi-diagonal matrices if and only if six certain commutators vanish.

The question of simultaneous quasi-diagonalizability has a physical application in unambiguous discrimination of quantum states (see the next paragraph). In that context, it is sufficient to deal with positive semi-definite operators with non-overlapping supports (the support of an operator is the orthocomplement of its kernel). As we will show, this restriction leads to simpler commutator criteria. In this paper we will give a constructive proof that, given two self-adjoint operators with non-overlapping supports, they have a common block diagonal structure of dimension 2, if and only if a set of only three commutators vanishes. These commutators are also easier to calculate than those given in [2], as the latter are of maximal order 7, while the former are of maximal order 5.

Unambiguous state discrimination (USD) is a strategy for distinguishing non-orthogonal quantum states without being allowed to make an error. As it is impossible to discriminate non-orthogonal quantum states with unit probability, the measurement has to have inconclusive outcomes. The optimal USD strategy is the one that maximizes the success probability (i.e., minimizes the probability to get an inconclusive result). A different possibility to discriminate quantum states is called *minimum error discrimination*, where one minimizes the probability of making an error in the state identification.

In this contribution we want to focus onto the first strategy, namely unambiguous state discrimination. For two density operators, ρ_1 and ρ_2 , acting on the Hilbert space \mathcal{H} of finite dimension, this task is described by a positive operator-valued measure (POVM) on \mathcal{H} , consisting of three positive operators, E_1 , E_2 and $E_?$, with $E_1 + E_2 + E_? = \mathbb{1}$. In order to make the discrimination unambiguous, the probability of wrong identification must vanish, i.e. $\text{tr}(E_1\rho_2) = 0$ and $\text{tr}(E_2\rho_1) = 0$. It is natural to allow ρ_1 and ρ_2 to have *a priori* probabilities p_1 and p_2 , respectively, where $p_1 > 0$, $p_2 > 0$, and $p_1 + p_2 = 1$. The open problem in USD is to find a POVM $\{E_1, E_2, E_?\}$ which maximizes the success probability $p_{\text{succ}} = p_1 \text{tr}(E_1\rho_1) + p_2 \text{tr}(E_2\rho_2)$.

While the optimal solution for minimum error discrimination of two mixed states is already known for more than three decades [3], the optimal solution for unambiguous state discrimination has been found only for the pure state case [4] and certain special cases of mixed states [5–13]. A partial solution for unambiguous discrimination of mixed states is provided via the reductions of the density operators by the space where perfect and/or no USD is possible [8]. Otherwise, known optimal USD measurements for mixed states mainly belong to the class, where the problem can be decomposed into several pure state discrimination tasks [5, 9, 11]. A general representation of such states was recently discussed by Bergou *et al* [11].

It is not obvious how to decide whether the given density operators possess such a structure. In this contribution we present a method that allows us to systematically identify if the optimal USD of two mixed states can be simplified to the pure state task.

The paper is organized as follows. In section 2 we introduce the concept of common block-diagonal structures of two operators. We specifically consider the case of two-dimensional blocks, as the optimal measurement in two dimensions is well known. Simple commutator relations are presented to check for the existence of such a structure. In section 3 we discuss whether the block structures are preserved by the reductions. Finally, we study the example of unambiguous state comparison [7, 9, 14–16] to illustrate the power of the commutator test.

2. Block-diagonal structures

2.1. Independent orthogonal subspaces in USD

In [5] Bennett *et al* analyzed the parity check for a string of qubits, i.e., the question whether a sequence composed of states that are either $|\psi_0\rangle$ or $|\psi_1\rangle$, with $0 < |\langle\psi_0|\psi_1\rangle| < 1$, contains

an even or odd number of occurrences of $|\psi_1\rangle$. This task is equivalent to the unambiguous discrimination of two certain mixed states. After a suitable (symmetric) choice of a basis these mixed states turned out to share the same block-diagonal shape, with each block \blacksquare symbolizing a 2×2 matrix:

$$\rho_1 = \begin{pmatrix} \blacksquare & & & \\ & \blacksquare & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \blacksquare & & & \\ & \blacksquare & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}. \quad (1)$$

The authors of [5] argued that due to this structure an optimal solution to the discrimination problem can be obtained by the simple composition of the optimal solutions in each block. The optimal solution in two dimensions is known, since only in the case of two pure states the solution is not obvious and this case was solved by Jaeger and Shimony [4].

Our aim is to provide a systematic method for finding such structures. We start with a formal definition of a block-diagonal structure: For a set of operators \mathcal{O} , a *common block-diagonal structure* (CBS) is a projection-valued measure $\{\Pi_k\}$ such that all operators in \mathcal{O} commute with any Π_k . In other words, if the operators in \mathcal{O} have a CBS, they can be simultaneously decomposed in orthogonal subspaces, and a von-Neumann measurement $\{\Pi_k\}$ projects onto these subspaces. Having the measurement outcome ‘ k ’, the support of the states is reduced to $\Pi_k\mathcal{H}$ (the image of Π_k). Thus one can focus on performing the optimal measurement in this subspace.

A common block-diagonal structure is *at most n -dimensional* if the rank of all Π_k is at most n . In particular, the existence of an at most one-dimensional CBS for a set \mathcal{O} of normal operators (a normal operator is an operator that commutes with its adjoint) is equivalent to the existence of a common basis, in which all operators in \mathcal{O} are diagonal. It is well known (cf, e.g., chapter IX, theorem 11 in [17]) that for normal operators this is possible if and only if all operators in \mathcal{O} mutually commute. We will present a commutator criterion to verify whether two operators have an at most *two-dimensional* CBS (2D-CBS). This criterion, which is simpler (from an operational point of view) than the one introduced by Laffey [2], is valid in the case of non-overlapping support only, but is sufficiently general in order to detect any two-dimensional block structure in the case of USD.

2.2. Diagonalizing Jordan bases: definition and existence

Let us first relate the idea of a 2D-CBS to a concept that is widely used in the analysis of USD, namely the concept of *Jordan (or canonical) bases* of subspaces (cf, e.g., [18]): let P_A and P_B be self-adjoint projectors. Then by virtue of the singular value decomposition, one can find orthonormal bases $\{|\alpha_i\rangle\}$ of $P_A\mathcal{H}$ and $\{|\beta_j\rangle\}$ of $P_B\mathcal{H}$, such that

$$\langle \alpha_i | \beta_j \rangle \equiv \langle \alpha_i | P_A P_B | \beta_j \rangle = 0 \quad \text{for } i \neq j, \quad (2a)$$

while for $i \leq \min\{\text{rank } P_A, \text{rank } P_B\}$,

$$\langle \alpha_i | \beta_i \rangle \equiv \langle \alpha_i | P_A P_B | \beta_i \rangle \equiv \cos \vartheta_i \geq 0 \quad (2b)$$

for some $0 \leq \vartheta_i \leq \pi/2$. The bases $\{|\alpha_i\rangle\}$ and $\{|\beta_j\rangle\}$ are called *Jordan bases* of the subspaces $P_A\mathcal{H}$ and $P_B\mathcal{H}$ and $\{\vartheta_i\}$ are the corresponding (unique) *Jordan angles*. The first equation expresses the bi-orthogonality of the Jordan bases. Note that in the case of degenerate Jordan angles (i.e., not all Jordan angles are different) or if $|\text{rank } P_A - \text{rank } P_B| \geq 2$, the Jordan bases are not unique.

For the analysis of USD, it turns out to be fruitful to consider density operators, which are diagonalized by a pair of Jordan bases [11]. For two normal operators A and B , *diagonalizing*

Jordan bases are Jordan bases of $\text{supp } A$ and $\text{supp } B$, which diagonalize A and B , respectively. Of course, such diagonalizing Jordan bases do not always exist. As mentioned in [19], the existence of such bases implies the presence of a 2D-CBS, since the pairs $\{|\alpha_i\rangle, |\beta_i\rangle\}$ span mutually orthogonal two-dimensional subspaces. However, the converse is in general not true. It is possible that already in two dimensions no pair of diagonalizing Jordan bases exists. Consider the positive semi-definite matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (3)$$

Then up to some complex phases, the only orthonormal basis of $\text{supp } A$ that diagonalizes A is the canonical basis $\{(1, 0), (0, 1)\}$ while $\text{supp } B$ is spanned by $(1, 1)$. But $(1, 1)$ is orthogonal to neither $(1, 0)$ nor $(0, 1)$, i.e., no diagonalizing Jordan bases exist.

The exact relation between 2D-CBS and diagonalizing Jordan bases is given by the following

Lemma 1. *Let A and B be normal operators acting on \mathcal{H} . Then diagonalizing Jordan bases of A and B can be found if and only if a 2D-CBS of A and B exists and $[A, ABA] = 0$ and $[B, BAB] = 0$.*

Proof. Assume that diagonalizing Jordan bases of A and B exist. Then their structure readily provides an appropriate 2D-CBS. Furthermore, by writing A and B in diagonalizing Jordan bases, i.e., $A = \sum_i a_i |\alpha_i\rangle\langle\alpha_i|$ and $B = \sum_j b_j |\beta_j\rangle\langle\beta_j|$, and using equations (2), it is easy to verify that $[A, ABA] = 0$ and $[B, BAB] = 0$ holds.

For the contrary it is enough to prove the assertion in each subspace $\Pi_k \mathcal{H}$, where $\{\Pi_k\}$ is a 2D-CBS of A and B . Since A and B commute with all projectors Π_k , in each subspace the operators $A_k \equiv \Pi_k A \Pi_k$ and $B_k \equiv \Pi_k B \Pi_k$ are again normal. First suppose that A_k has a maximal rank, i.e., rank 2. Since A_k has full rank in $\Pi_k \mathcal{H}$, the condition $0 = \Pi_k [A, ABA] \Pi_k = A_k [A_k, B_k] A_k$ is equivalent to $\Pi_k [A_k, B_k] \Pi_k \equiv [A_k, B_k] = 0$, i.e., both operators can be diagonalized simultaneously and hence in particular diagonalizing Jordan bases exist. (An analogous argument holds if B_k has a maximal rank.) The remaining non-trivial case is that both operators have rank 1, in which case the diagonalizing Jordan bases are given by the vector spanning the support of each operator. \square

Note that commutators of the form $[A, AXA]$ can always be rewritten as $A[A, X]A$, i.e., in the above lemma one could equivalently write the conditions $A[A, B]A = 0$ and $B[A, B]B = 0$.

2.3. Construction of diagonalizing Jordan bases

It is a simple observation that if diagonalizing Jordan bases for two normal operators A and B exist, then necessarily all commutators of the structure $[A, ABA]$, $[A, AB^2A]$ and so forth vanish (see the proof of lemma 1). In the following lemma we will state that certain of these commutators already suffice to explicitly construct a pair of diagonalizing Jordan bases.

Lemma 2. *Let A and B be self-adjoint operators on \mathcal{H} with $[A, ABA] = 0$, $[A, AB^2A] = 0$ and $[B, BA^2B] = 0$. Furthermore, denote by $\{|k\rangle\}$ an orthogonal basis of $\text{supp } A$ which simultaneously diagonalizes A , ABA and AB^2A .*

Then there exists vectors $\{|v\rangle\}$, such that (up to normalization) $\{A|k\rangle\}$ and $\{BA|k\rangle\} \cup \{|v\rangle\}$ are diagonalizing Jordan bases of A and B .

Proof. First note that all vectors $BA|k\rangle$ are mutually orthogonal (or trivial), since the basis $\{|k\rangle\}$ diagonalizes $ABBA$. Now consider the following expression:

$$\begin{aligned} w_k B(BA|k\rangle) &= BBA(ABA|k\rangle) \\ &= BA^2BBA|k\rangle \\ &= v_k BA|k\rangle, \end{aligned} \tag{4}$$

where w_k denotes the eigenvalue of ABA for $|k\rangle$ and v_k denotes the eigenvalue of AB^2A for $|k\rangle$. In the second step we used $[B, BA^2B] = 0$. The right-hand side can only vanish if $BA|k\rangle = 0$. Hence due to equation (4), $BA|k\rangle \in \text{supp } B$ is either trivial or is an eigenvector of B . Furthermore, one readily finds eigenvectors $|\nu\rangle \in \text{supp } B$ of B that complete the orthogonal basis of $\text{supp } B$. These vectors are also orthogonal to all $A|k\rangle$, since by construction $b_\nu \langle \nu|A|k\rangle = \langle \nu|BA|k\rangle = 0$, where $b_\nu \neq 0$ is the eigenvalue of B for $|\nu\rangle$. It remains to verify that $\{A|k\rangle\}$ and $\{BA|k\rangle\}$ are bi-orthogonal. But this follows from the fact that $\{|k\rangle\}$ diagonalizes ABA . \square

Note that it is straightforward to extend this lemma to normal operators. However, we are mainly interested in application for USD and hence specialize the results of this section in the following form:

Theorem. For two self-adjoint A and B operators on a Hilbert space of finite dimension with $\text{supp } A \cap \text{supp } B = \{0\}$ the following statements are equivalent: (i) A and B have a 2D-CBS; (ii) diagonalizing Jordan bases of A and B exist; (iii) $[A, ABA] = 0$, $[B, BA^2B] = 0$ and $[A, AB^2A] = 0$.

Proof. Remember that (ii) \Rightarrow (i) follows from the structure of Jordan bases (see lemma 1), and also (ii) \Rightarrow (iii) is a consequence of the properties of Jordan bases (i.e., that all commutators of the structure $[A, ABA]$, $[A, AB^2A]$ and so forth vanish). The implication (iii) \Rightarrow (ii) was proven in lemma 2. It remains to show that from (i) follows (ii). Due to lemma 1 this reduces to showing that $[A, ABA] = 0$ and $[B, BAB] = 0$ for the case where (i) holds and $\text{supp } A \cap \text{supp } B = \{0\}$. The condition of non-overlapping supports implies, together with (i), that $\text{rank}(A_k) + \text{rank}(B_k) \leq 2$, where $A_k = \Pi_k A \Pi_k$ and $B_k = \Pi_k B \Pi_k$, and $\{\Pi_k\}$ is a 2D-CBS of A and B . If either $\text{rank}(A_k)$ or $\text{rank}(B_k)$ is zero, the commutators $[A_k, A_k B_k A_k]$ and $[B_k, B_k A_k B_k]$ vanish trivially. They are also equal to zero for the remaining case of $\text{rank}(A_k) = 1 = \text{rank}(B_k)$. \square

As soon as the supports of A and B overlap, in general, none of the commutators in the above theorem vanishes. But in such a situation one can make use of the fact that in two dimensions, the square of all commutators of the form $[A_k, B_k]$, $[A_k, B_k^2]$ and so forth is proportional to the identity operator. Laffey [2] showed that for positive operators the following set of commutators, given below, are already sufficient to prove the existence of a 2D-CBS.

Two positive semi-definite operators A and B have a 2D-CBS if and only if [2]

$$\begin{aligned} [[A, B]^2, A] &= 0, & [[B, A]^2, B] &= 0, \\ [[A, B^2]^2, A] &= 0, & [[B, A^2]^2, B] &= 0, \\ [[A^2, B]^2, A] &= 0, & [[B^2, A]^2, B] &= 0. \end{aligned} \tag{5}$$

3. Application to USD

We now want to apply the above analysis to unambiguous discrimination of two mixed states ρ_1 and ρ_2 . We denote the combination of the density operator and the according *a priori* probability by $\gamma_\mu = p_\mu \rho_\mu$, such that $\text{tr} \gamma_\mu < 1$ ($\mu = 1, 2$). For technical reasons (see the map τ_0 below) we also allow that the *a priori* probabilities do not sum up to 1, $\text{tr}(\gamma_1) + \text{tr}(\gamma_2) \leq 1$.

3.1. Preservation of block structures under reduction of USD

In the above theorem the density operators need to satisfy the condition $\text{supp} \gamma_1 \cap \text{supp} \gamma_2 = \{0\}$, which in general is not the case. The first reduction theorem in [8], however, shows how to reduce any USD problem to that specific form. But one could imagine that this reduction might destroy an already present 2D-CBS, so that the combination of the first reduction theorem together with the above theorem would fail to detect certain block-diagonal structures. As we will see here, this is not the case and the application of any of the reductions in [8] preserves any CBS.

We repeat the reductions of [8] in the language of projectors. For a pair of positive semi-definite operators (γ_1, γ_2) , let τ_0 be the (nonlinear) mapping

$$\tau_0: (\gamma_1, \gamma_2) \mapsto (\gamma_1^0, \gamma_2^0), \quad (6)$$

where γ_μ^0 (with $\mu = 1, 2$) is the projection of γ_μ onto $(\ker \gamma_1 + \ker \gamma_2)$. In a similar fashion we define $\tau_\nu: (\gamma_1, \gamma_2) \mapsto (\gamma_1^\nu, \gamma_2^\nu)$ (with $\nu = 1, 2$) where

$$\gamma_\mu^\nu = P_\nu \gamma_\mu P_\nu + (\mathbb{1} - P_\nu) \gamma_\mu (\mathbb{1} - P_\nu). \quad (7)$$

Here, P_1 is the self-adjoint projector onto $(\ker \gamma_1 + \text{supp} \gamma_2)$ and P_2 the projection onto $(\ker \gamma_2 + \text{supp} \gamma_1)$. The reduction theorems in [8] now read as follows.

For $\tau \in \{\tau_0, \tau_1, \tau_2\}$, the pair (γ_1, γ_2) and the reduced pair $\tau(\gamma_1, \gamma_2)$ can be unambiguously discriminated with the same success probability [8].

What is relevant for our considerations is the fact that no reduction can destroy any CBS, i.e., a CBS $\{\Pi_k\}$ of (γ_1, γ_2) is also a CBS of $\tau(\gamma_1, \gamma_2)$ for all $\tau \in \{\tau_0, \tau_1, \tau_2\}$. In order to see this, it is enough to show that any of the projectors P_0, P_1 and P_2 (with P_0 denoting the projector onto $\ker \gamma_1 + \ker \gamma_2$) commutes with all Π_k . But this follows from the fact that the range of each of the projectors is the support of an operator that commutes with all Π_k (namely, $P_0 \mathcal{H} = \text{supp}(2\mathbb{1} - G_1 - G_2)$, $P_1 \mathcal{H} = \text{supp}(\mathbb{1} - G_1 + G_2)$ and $P_2 \mathcal{H} = \text{supp}(\mathbb{1} - G_2 + G_1)$, where G_μ is the projector onto $\text{supp} \gamma_\mu$). Note, however, in contrast, that a CBS of $\tau(\gamma_1, \gamma_2)$ is not necessarily a CBS of (γ_1, γ_2) , thus a reduction may give rise to new block-diagonal structures.

In order to check for a 2D-CBS it is necessary to first apply the reduction τ_0 . If the reductions τ_1 and τ_2 are—from an operational point of view—feasible, then it is also worthwhile to apply those, since new 2D-CBS may arise.

3.2. Example: state comparison

We consider a special case of unambiguous state comparison ‘two out of N ’ as defined in [16]. A source emits pure states $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$, each of which appears with equal *a priori* probability $\frac{1}{N}$. We further assume that all states have the same (real) mutual overlap, $\langle \psi_i | \psi_j \rangle = \cos \vartheta$ for $i \neq j$. Given two of these pure states, the aim is to decide unambiguously

whether the states are identical or not. This task is equivalent to the discrimination of

$$\gamma_1 = \frac{1}{N^2} \sum_{k=1}^N |\psi_k \psi_k\rangle \langle \psi_k \psi_k|, \tag{8}$$

$$\gamma_2 = \frac{1}{N^2} \sum_{k \neq l}^N |\psi_k \psi_l\rangle \langle \psi_k \psi_l|. \tag{9}$$

From the definition it follows that $\text{supp } \gamma_1 \cap \text{supp } \gamma_2 = \{0\}$. Thus we can directly apply the theorem of section 2.3, i.e., we test whether it is true that $[\gamma_1, \gamma_1 \gamma_2 \gamma_1] = 0$, $[\gamma_1, \gamma_1 \gamma_2^2 \gamma_1] = 0$ and $[\gamma_2, \gamma_2 \gamma_1^2 \gamma_2] = 0$. For the first two commutators, it is sufficient to verify that $\omega_{kl} \equiv \langle \psi_k \psi_k | [\cdot \cdot] | \psi_l \psi_l \rangle = 0$ for any k and l . Here, $[\cdot \cdot]$ stands for any of the first two commutators. Obviously we have $\omega_{kl} = -(\omega_{lk})^*$ for all k and l , and since all overlaps are real, $\omega_{kk} = 0$. Due to the high symmetry, all ω_{kl} with $k \neq l$ must be equal. In particular, $\omega_{kl} = \omega_{lk} = -(\omega_{kl})^*$, and again due to reality of the overlaps, $\omega_{kl} = 0$ must hold.

It remains to test whether $[\gamma_2, \gamma_2 \gamma_1^2 \gamma_2] = 0$. This is equivalent to showing that $\gamma_2 [\gamma_2 + \gamma_1, \gamma_1^2] \gamma_2 = 0$ or to showing that

$$\gamma_2 (\gamma_2 + \gamma_1) \gamma_1^2 \gamma_2 = \sum_{i,j;p,q} |\psi_i \psi_j\rangle \langle \psi_p \psi_q| A_{ij,pq} \tag{10}$$

is self-adjoint. For $i \neq j$ and also $p \neq q$, we have

$$A_{ij,pq} = \sum_{k,l,n,m} c_{ik} c_{jl} c_{kn} c_{ln} c_{nm}^2 c_{mp} c_{mq}, \tag{11}$$

with $c_{ij} \equiv \langle \psi_i | \psi_j \rangle = \cos \vartheta + (1 - \cos \vartheta) \delta_{ij}$. Otherwise, $A_{ij,pq} = 0$. First we find

$$\sum_k c_{ik} c_{kn} \propto \delta_{in} + \mu, \tag{12}$$

with some constant μ . Also, for $p \neq q$,

$$\sum_m c_{nm}^2 c_{mp} c_{mq} \propto \delta_{nq} + \delta_{np} + \sigma, \tag{13}$$

where σ is another constant. Hence for $i \neq j$ and $p \neq q$ we have

$$\begin{aligned} A_{ij,pq} &\propto \sum_n (\delta_{in} + \mu)(\delta_{jn} + \mu)(\delta_{np} + \delta_{nq} + \sigma) \\ &\propto \delta_{ip} + \delta_{iq} + \delta_{jp} + \delta_{jq} + \text{const}. \end{aligned} \tag{14}$$

In particular, $A_{ij,pq} = A_{pq,ij} \equiv (A_{pq,ij})^*$ holds, which demonstrates that $\gamma_2 (\gamma_2 + \gamma_1) \gamma_1^2 \gamma_2$ is self-adjoint and therefore $\gamma_2 [\gamma_2 + \gamma_1, \gamma_1^2] \gamma_2 = 0$.

Thus we have shown that the symmetric state comparison ‘two out of N ’ can be reduced to pure state discrimination. Note that this statement is in general not true for state comparison ‘ C out of N ’, with $C > 2$, i.e., the question whether C states taken from a set of N states (with equal overlaps) are identical or not. In this case the third commutator does not vanish before the reductions, and the corresponding state discrimination problem is not necessarily simplified to the pure state case.

4. Conclusions

In many practical situations of unambiguous state discrimination (USD) the pair of states that one wants to discriminate has a high symmetry which naturally gives rise to a two-dimensional common block-diagonal structure (2D-CBS) [5, 9, 11]. In this situation the

optimal USD measurement has the very same 2D-CBS [11], where each block basically is given by the pure state solution of Jaeger and Shimony [4].

Here, we provided a tool to systematically identify whether a given USD task possesses such a structure. With the commutator relations presented in this paper it is easy to test whether a 2D-CBS for two self-adjoint operators with non-overlapping support exists. In order to derive these commutator relations, we studied the connection between the existence of a 2D-CBS and of diagonalizing Jordan bases. This also led to an explicit construction procedure for such bases.

We showed that the reduction method [8] for USD can only generate, but not destroy a 2D-CBS. Thus, applying the reductions as a first step ensures that the condition of non-overlapping support of the two operators is fulfilled.

We demonstrated the strength of the simple commutator relations by considering unambiguous state comparison [7, 9, 14–16], where it is easy to show that in completely symmetric situations for the specific case ‘two out of N ’ a 2D-CBS exists.

Outlook. Note that the commutator relations in the theorem of section 2.3 are not symmetric in both operators (i.e., the missing commutator $[BAB, B]$ already vanishes). It would be interesting to understand the reason for this asymmetry. Furthermore, it would be useful to extend this concept to be applicable to more than two operators and also to the detection of larger block-diagonal structures (with respect to USD, e.g., four-dimensional structures would be interesting). In order to be operational, this would mean to extend the work by Watters [1] and Shapiro [20] (generalization to blocks of arbitrary dimension) and finding a *finite* set of commutators with possibly low order.

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